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# FILTRATIONS AND CANONICAL COORDINATES ON NILPOTENT LIE GROUPS

BY

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ABSTRACT. Let $\mathfrak{g}$ be a finite-dimensional nilpotent Lie algebra over a field of characteristic zero. Introducing the notion of a positive, decreasing filtration $\mathfrak{F}$ on $\mathfrak{g}$, the paper studies the multiplicative structure of the universal enveloping algebra $U(\mathfrak{g})$, and also transformation laws between $\mathfrak{F}$-canonical coordinates of the first and second kind associated with the Campbell-Hausdorff group structure on $\mathfrak{g}$. The basic technique is to exploit the duality between $U(\mathfrak{g})$ and $S(\mathfrak{g}^*)$, the symmetric algebra of $\mathfrak{g}^*$, making use of the filtration $\mathfrak{F}$. When the field is the complex numbers, the preceding results, together with the Cauchy estimates, are used to obtain estimates for the structure constants for $U(\mathfrak{g})$. These estimates are applied to construct a family of completions $U(\mathfrak{g})_{\mathfrak{M}}$ of $U(\mathfrak{g})$, on which the corresponding simply-connected Lie group $G$ acts by an extension of the adjoint representation.

**Introduction.** Let $G$ be a connected and simply-connected Lie group with Lie algebra $\mathfrak{g}$, and let $\{X_i: 1 \leqslant i \leqslant d\}$ be a basis for $\mathfrak{g}$. Then there are "canonical coordinates" $\{\xi_i\}$ and $\{\eta_i\}$ of the "first kind" and "second kind", respectively, on $G$ defined by this basis [2, III.4.3]. These functions, which are defined on a neighborhood of the identity, are related by

$$\exp(\xi_1 X_1 + \cdots + \xi_d X_d) = \exp \eta_1 X_1 \cdots \exp \eta_d X_d, \tag{0.1}$$

where $\exp: \mathfrak{g} \to G$ is the exponential map. If $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$, then the given basis for $\mathfrak{g}$ also defines bases $\{X(\alpha): \alpha \in \mathbf{N}^d\}$ and $\{X^\alpha: \alpha \in \mathbf{N}^d\}$ for $U(\mathfrak{g})$ (where $X^\alpha = X_1^{\alpha_1} \cdots X_d^{\alpha_d}$ and $X(\alpha)$ is the symmetrization of $X^\alpha$). The goal of this paper is to study the multiplicative structure of $U(\mathfrak{g})$, as expressed in terms of these bases, by using the duality between $U(\mathfrak{g})$ and polynomials in the canonical coordinates.

We shall restrict attention to *nilpotent* Lie algebras $\mathfrak{g}$, and carry out all constructions relative to a given positive, decreasing filtration $\mathfrak{F}$ on $\mathfrak{g}$ (e.g. $\mathfrak{F}$ = descending central series). Following G. Birkhoff [1], we extend $\mathfrak{F}$ to a decreasing filtration of $U(\mathfrak{g})$ by ideals of finite codimension in §1. This gives

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"two-sided" vanishing conditions on the structure constants for the multiplication on $U(\mathfrak{g})$, and allows us to define a "formal noncommutative power series" completion $[U(\mathfrak{g})]_{\mathfrak{F}}$ of $U(\mathfrak{g})$.

In §2 we pass by duality to an increasing filtration on the commutative algebra $\mathfrak{P}$ of polynomial functions on $\mathfrak{g}$. The multiplication on $U(\mathfrak{g})$ dualizes to a comultiplication on $\mathfrak{P}$ which preserves the filtration.

It is well known that if $\{X_i\}$ is a *Jordan-Hölder* basis (i.e. the subspaces $\mathfrak{h}_i = \text{span}\{X_k: k \geqslant i\}$ are ideals in $\mathfrak{g}$), then $\{\xi_i\}$ and $\{\eta_i\}$, defined by (0.1), give global coordinates for $G$ and are related by a polynomial transformation $\phi$. If we make the stronger requirement that $\{X_i\}$ be an *$\mathfrak{F}$-basis* (definition in §1), then we can say more about $\phi$: In §3 we construct a simply-connected nilpotent Lie group $M = M(\mathfrak{F})$, faithfully represented as a group of locally unipotent automorphisms of $\mathfrak{P}$, and in §4 we prove that $\phi \in M$. By duality, we are then able to study the transformation from the basis $\{X^\alpha\}$ to the basis $\{X(\alpha)\}$ of $U(\mathfrak{g})$.

The original motivation for this paper was the construction of algebras of "differential operators of infinite order" on $G$, obtained by completing $U(\mathfrak{g})$ relative to suitable locally convex topologies (cf. [5]). We present a class of such algebras in §6, using estimates for the structure constants for $U(\mathfrak{g})$ obtained in §5. The construction of these algebras is considerably simpler and more general than in [5]. In future work we plan to relate the representation theory of these algebras to that of the group $G$ and the work of Treves and collaborators on "hyperdifferential operators" (cf. [11]).

This paper continues the study of nilpotent Lie algebras and groups via filtrations started in [6] (cf. [7, Chapter I and Appendix]).

Since we are considering only nilpotent Lie algebras, the restriction to real algebras is unnecessary. In the purely algebraic part of the paper (§§1–4), the coefficient field is any field $\mathbf{F}$ of characteristic zero. In §§5–6, we assume for the purpose of making estimates that $\mathbf{F} = \mathbf{C}$ (i.e. in the original context of a real Lie group $G$, we pass to the *complexified* universal enveloping algebra).

We use the customary notations of $\mathbf{N}$ for the nonnegative integers, $\mathbf{Q}$ for the rational numbers, and $\mathbf{C}$ for the complex numbers.

**1. Filtrations on $\mathfrak{g}$ and $U(\mathfrak{g})$.** Let $\mathbf{F}$ be a field of characteristic zero, and let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbf{F}$. Denote by $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$.

DEFINITION. A *positive filtration* $\mathfrak{F}$ on $\mathfrak{g}$ is a decreasing chain of subspaces $\{\mathfrak{g}_n\}_{n \geqslant 1}$, such that

$$\mathfrak{g} = \mathfrak{g}_1 \supseteq \mathfrak{g}_2 \supseteq \cdots \supseteq \mathfrak{g}_l \supseteq \mathfrak{g}_{l+1} = 0, \qquad [\mathfrak{g}_m, \mathfrak{g}_n] \subseteq \mathfrak{g}_{m+n}.$$

The *length* of $\mathfrak{F}$ is the smallest integer $l$ such that $\mathfrak{g}_{l+1} = 0$.

Note that there exists a positive filtration on $\mathfrak{g}$ if and only if $\mathfrak{g}$ is nilpotent, with the shortest filtration being the descending central series ($\mathfrak{g}_{n+1} = [\mathfrak{g}, \mathfrak{g}_n]$).

Fix a positive filtration $\mathfrak{F}$, and for $X \in \mathfrak{g}$, $X \neq 0$, define the *$\mathfrak{F}$-weight* by

$$w(X) = \max\{n: X \in \mathfrak{g}_n\}.$$

We shall say that a basis $\{X_i: 1 \leq i \leq d\}$ for $\mathfrak{g}$ is an *$\mathfrak{F}$-basis* if $w(X_i) \leq w(X_{i+1})$ and

$$\mathfrak{g}_n = \operatorname{span}\{X_i: w(X_i) \geq n\},$$

for $n = 1, 2, \ldots, l$. The numbers $w_i = w(X_i)$ are then independent of the choice of $\mathfrak{F}$-basis. If $\alpha \in \mathbf{N}^d$ ($d = \dim \mathfrak{g}$), set

$$|\alpha| = \sum \alpha_i, \qquad w(\alpha) = \sum w_i \alpha_i.$$

Using the basis $\{X_i\}$ for $\mathfrak{g}$, we can construct two bases for $U(\mathfrak{g})$. Namely, given $\alpha \in \mathbf{N}^d$, set $n = |\alpha|$ and define

$$X^\alpha = X_1^{\alpha_1} \cdots X_d^{\alpha_d}, \qquad X(\alpha) = (n!)^{-1}(\partial/\partial t)^\alpha (t_1 X_1 + \cdots + t_d X_d)^{|\alpha|},$$

where the products are in $U(\mathfrak{g})$ and

$$(\partial/\partial t)^\alpha = (\partial/\partial t_1)^{\alpha_1} \cdots (\partial/\partial t_d)^{\alpha_d}.$$

By the Poincaré-Birkhoff-Witt (PBW) theorem, $\{X(\alpha): \alpha \in \mathbf{N}^d\}$ and $\{X^\alpha: \alpha \in \mathbf{N}^d\}$ are bases for $U(\mathfrak{g})$, which we shall call bases of the *first* and *second* kind, respectively (cf. [3, §2.1]). Note that $X(\alpha)$ is the "symmetrization" of $X^\alpha$.

The multiplicative structure of $U(\mathfrak{g})$ can then be expressed in terms of these bases by the equations

$$(\alpha!\beta!)^{-1} X(\alpha) X(\beta) = \sum (\gamma!)^{-1} C_\gamma^{\alpha\beta} X(\gamma), \tag{1.1}$$

$$(\alpha!\beta!)^{-1} X^\alpha X^\beta = \sum (\gamma!)^{-1} K_\gamma^{\alpha\beta} X^\gamma. \tag{1.2}$$

We shall call $\{C_\gamma^{\alpha\beta}\}$ and $\{K_\gamma^{\alpha\beta}\}$ the *structure constants* (of the first and second kind, respectively) for $U(\mathfrak{g})$.

PROPOSITION 1.1. *If $\{X_i\}$ is an $\mathfrak{F}$-basis, then the structure constants satisfy*

$$C_\gamma^{\alpha\beta} = K_\gamma^{\alpha\beta} = 0 \tag{1.3}$$

*if either $|\gamma| > |\alpha| + |\beta|$ or $w(\gamma) < w(\alpha) + w(\gamma)$.*

REMARK. If the map $X_i \to w_i X_i$ extends to a *derivation* of $\mathfrak{g}$, then there is an associated action of the multiplicative group $\mathbf{F}^\times$ on $U(\mathfrak{g})$, defined by $t \cdot X^\alpha = t^{w(\alpha)} X^\alpha$. It follows that in this case (1.3) holds whenever $w(\gamma) \neq w(\alpha) + w(\beta)$. In general, however, there will not exist an $\mathfrak{F}$-basis with this property, since $\mathfrak{g}$ does not always have a one-parameter group of dilating automorphisms (cf. [4] and [7, §I.3.2]).

PROOF. The vanishing conditions (1.3) follow from the existence of *two* filtrations on $U(\mathfrak{g})$. Namely, let

$$U_n = \operatorname{span}\{X^\alpha: |\alpha| \leqslant n\} = \operatorname{span}\{X(\alpha): |\alpha| \leqslant n\}$$

be the canonical *increasing* filtration of $U(\mathfrak{g})$. The property $U_m U_n \subseteq U_{m+n}$ then implies (1.3) when $|\alpha| + |\beta| < |\gamma|$.

In the other direction, if we define (following [1])

$$J_n = \operatorname{span}\{X^\alpha: w(\alpha) \geqslant n\},$$

then the PBW theorem and the filtration condition $[\mathfrak{g}_m, \mathfrak{g}_n] \subseteq \mathfrak{g}_{m+n}$ imply that

$$J_n = \sum \mathfrak{g}_{n_1} \cdots \mathfrak{g}_{n_k} \qquad (n_1 + \cdots + n_k \geqslant n), \tag{1.4}$$

(of course, (1.4) is not a direct sum). This makes it evident that

$$J_m J_n \subseteq J_{m+n}. \tag{1.5}$$

From the definition, we see that

$$U(\mathfrak{g}) = J_0 \supseteq J_1 \supseteq \cdots. \tag{1.6}$$

Obviously we also have

$$\bigcap_n J_n = 0. \tag{1.7}$$

Thus $\{J_n\}$ is a *decreasing*, separated filtration on $U(\mathfrak{g})$. By (1.4) it is independent of the choice of $\mathfrak{F}$-basis, and

$$J_n = \operatorname{span}\{X(\alpha): w(\alpha) \geqslant n\}.$$

The PBW theorem and the filtration property (1.5) gives (1.3) when $w(\alpha) + w(\beta) > w(\gamma)$. This proves the proposition.

Using the filtration $\{J_n\}$ introduced above, we define a topology on $U(\mathfrak{g})$ by letting the sets $\{a + J_n\}$ be a basis for the neighborhoods of $a \in U(\mathfrak{g})$. The filtration condition (1.5) then shows that miltiplication is jointly continuous in this topology, so that $U(\mathfrak{g})$ becomes a topological algebra. Denote by $[U(\mathfrak{g})]_{\mathfrak{F}}$ the completion of $U(\mathfrak{g})$ in this uniformity (cf. [9, Chapter X]). The algebraic operations on $U(\mathfrak{g})$ extend by continuity to $[U(\mathfrak{g})]_{\mathfrak{F}}$. An easy argument using the PBW theorem establishes the following explicit realization of $[U(\mathfrak{g})]_{\mathfrak{F}}$:

PROPOSITION 1.2. *The algebra* $[U(\mathfrak{g})]_{\mathfrak{F}}$ *is naturally isomorphic to the algebra of all formal series* $\Sigma a_\alpha X^\alpha$, *where* $\{X_i\}$ *is an* $\mathfrak{F}$*-basis for* $\mathfrak{g}$ *and the multiplication is defined using equations* (1.2).

REMARKS. (1) The product of the series $\Sigma(\alpha!)^{-1} a_\alpha X^\alpha$ and $\Sigma(\beta!)^{-1} b_\beta X^\beta$ is the series $\Sigma(\gamma!)^{-1} c_\gamma X^\gamma$, where

$$c_\gamma = \sum_{\alpha,\beta} K_\gamma^{\alpha\beta} a_\alpha b_\beta. \tag{1.8}$$

Note that by (1.3) the range of summation on the right side of (1.8) is finite.

(2) The analogue of Proposition 1.2, using the symmetrized basis $\{X(\alpha)\}$ instead of $\{X^\alpha\}$, also holds.

(3) The referee has pointed out that the uniform structure on $U(\mathfrak{g})$ introduced above is independent of the choice of positive filtration on $\mathfrak{g}$. Indeed, if $\mathfrak{F}'$ is another positive filtration of $\mathfrak{g}$, and $\{J_n'\}$ the corresponding filtration of $U(\mathfrak{g})$, then it is easy to verify that $J_n \subseteq J'_{[n/l]}$, where $l$ is the length of $\mathfrak{F}$. Hence by symmetry, we conclude that $\{J_n\}$ and $\{J_n'\}$ define the same set of neighborhoods of 0, and thus the same uniform structure, as asserted.

**2. Filtrations and comultiplication on** $S(\mathfrak{g}^*)$. Let $\mathfrak{g}^*$ be the vector space dual to $\mathfrak{g}$, and $S(\mathfrak{g}^*)$ the symmetric tensor algebra over $\mathfrak{g}^*$. We shall identify $S(\mathfrak{g}^*)$ with the algebra $\mathfrak{P}$ of polynomial functions on $\mathfrak{g}$, as usual. If $X \in \mathfrak{g}$, let $\partial_X$ be the unique derivation of $\mathfrak{P}$ such that

$$\partial_X(\xi) = \langle \xi, X\rangle, \qquad \xi \in \mathfrak{g}^*.$$

There is a unique bilinear pairing between $\mathfrak{P}$ and $U(\mathfrak{g})$ such that $\langle f, 1\rangle = f(0)$ and

$$\langle f, X^n\rangle = (\partial_X)^n f(0), \tag{2.1}$$

for $f \in \mathfrak{P}$ and $X \in \mathfrak{g}$. Given $\{X_i\}$, a basis for $\mathfrak{g}$, we set

$$\partial_i = \partial_{X_i}, \qquad \partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}.$$

Then this pairing is given by

$$\langle f, X(\alpha)\rangle = \partial^\alpha f(0), \tag{2.2}$$

in terms of the corresponding basis of the "first kind" for $U(\mathfrak{g})$. In particular, formula (2.2) and the PBW theorem imply that this pairing is nonsingular.

Suppose now that $\mathfrak{F}$ is a positive filtration on $\mathfrak{g}$, and let $\{J_n\}$ be the induced filtration on $U(\mathfrak{g})$, as in §1. We define

$$\mathfrak{P}_n = J_{n+1}^\perp = \{f \in \mathfrak{P} : \langle f, J_{n+1}\rangle = 0\}.$$

Let $\{X_i\}$ be an $\mathfrak{F}$-basis for $\mathfrak{g}$, and let $\{\xi_i\}$ be the dual basis for $\mathfrak{g}^*$. Then from (2.2) we see that

$$\begin{aligned}\mathfrak{P}_n &= \{f \in \mathfrak{P} : \partial^\alpha f(0) = 0 \text{ if } w(\alpha) > n\}\\ &= \operatorname{span}\{\xi^\alpha : w(\alpha) \leqslant n\},\end{aligned} \tag{2.3}$$

where $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d}$. This makes it evident that $\{\mathfrak{P}_n\}$ is an increasing filtration of the algebra $\mathfrak{P}$. It is clear from the definition of $\mathfrak{P}_n$ that the pairing between $U(\mathfrak{g})$ and $\mathfrak{P}$ extends by continuity to a pairing between $[U(\mathfrak{g})]_{\mathfrak{F}}$ and $\mathfrak{P}$.

Let $(U \otimes U)^*$ be the vector space dual to $U(\mathfrak{g}) \otimes U(\mathfrak{g})$. Via the pairing (2.1), we may consider $\mathfrak{P} \otimes \mathfrak{P}$ as a subspace of $(U \otimes U)^*$. We define a filtration on $\mathfrak{P} \otimes \mathfrak{P}$ by setting

$$(\mathcal{P} \otimes \mathcal{P})_n = \sum \mathcal{P}_k \otimes \mathcal{P}_m \qquad (k + m \leq n).$$

By duality, the multiplication on $U(\mathfrak{g})$ defines a linear map

$$\mu: \mathcal{P} \to (U \otimes U)^*,$$

such that

$$\langle \mu(f), S \otimes T \rangle = \langle f, ST \rangle \tag{2.4}$$

for $f \in \mathcal{P}$ and $S, T \in U(\mathfrak{g})$.

PROPOSITION 2.1. *The map $\mu$ is an algebra homomorphism from $\mathcal{P}$ to $\mathcal{P} \otimes \mathcal{P}$ and carries $\mathcal{P}_n$ into $(\mathcal{P} \otimes \mathcal{P})_n$.*

PROOF. Let $\{X_i\}$ be an $\mathcal{F}$-basis for $\mathfrak{g}$, and let the structure constants $C_\gamma^{\alpha\beta}$ be defined by (1.1). It then follows from (2.2) that

$$\mu(\xi^\gamma) = \sum_{\alpha,\beta} C_\gamma^{\alpha\beta} \xi^\alpha \otimes \xi^\beta. \tag{2.5}$$

In view of (2.3) and Proposition 1.1, we see that the right side of (2.5) is in $(\mathcal{P} \otimes \mathcal{P})_{w(\gamma)}$, so that $\mu$ maps $\mathcal{P}$ into $\mathcal{P} \otimes \mathcal{P}$ and preserves the filtrations.

It remains to show that $\mu(fg) = \mu(f)\mu(g)$. For this, we first observe that if $f \in \mathcal{P}$ and $X \in \mathfrak{g}$, then by Taylor's formula,

$$f(X) = \langle f, e^X \rangle, \tag{2.6}$$

where $e^X$ is defined by its power series as an element of $[U(\mathfrak{g})]_{\mathcal{F}}$. Identifying $\mathcal{P} \otimes \mathcal{P}$ with the polynomial functions on $\mathfrak{g} \times \mathfrak{g}$ as usual, we can evaluate $F \in \mathcal{P} \otimes \mathcal{P}$ in terms of the pairing by

$$F(X, Y) = \langle F, e^X \otimes e^Y \rangle.$$

When $F = \mu(f)$, this gives the formula

$$\mu(f)(X, Y) = \langle f, e^X e^Y \rangle,$$

for $f \in \mathcal{P}$ and $X, Y \in \mathfrak{g}$.

On the other hand, since $\mathfrak{g}$ is nilpotent, the Campbell-Hausdorff-Dynkin formula [9, Chapter X] shows that the formal series $e^X e^Y$ can be rearranged to be expressed as $e^Z$, where

$$Z = X + Y + \tfrac{1}{2}[X, Y] + \cdots$$

is a Lie polynomial in $X, Y$ with coefficients in $\mathbf{Q}$. Writing $Z = X * Y$, we can thus evaluate

$$\mu(f)(X, Y) = f(X * Y),$$

which shows that $\mu$ is multiplicative. Q.E.D.

We shall call $\mu$ the *comultiplication* on $\mathcal{P}$. Using the multiplicative property of $\mu$ and equation (2.5), one obtains the following formula expressing the structure constants of the first kind in terms of canonical coordinates of the first kind (cf. [8, Chapter VI, §4]):

COROLLARY 2.2. *Let the constants* $C_\gamma^{\alpha\beta}$ *be defined by* (1.1) *relative to an* $\mathcal{F}$*-basis* $\{X_i\}$ *for* $\mathfrak{g}$, *let* $\{\xi_i\}$ *be the dual basis for* $\mathfrak{g}^*$, *and define* $F_i = \mu(\xi_i)$. *Then*

$$C_\gamma^{\alpha\beta} = (\alpha!\beta!)^{-1}\partial_{(x)}^\alpha\partial_{(y)}^\beta F^\gamma(x,y)|_{x=y=0}, \tag{2.7}$$

*where* $F^\gamma = F_1^{\gamma_1}\cdots F_d^{\gamma_d}$.

**3. Automorphisms of** $S(\mathfrak{g}^*)$. In order to pass from the basis $\{X(\alpha)\}$ to the basis $\{X^\alpha\}$ for $U(\mathfrak{g})$, it will be useful to introduce a group of locally unipotent automorphisms of the algebra $S(\mathfrak{g}^*) = \mathcal{P}$. (Recall that a linear transformation $\phi$ on a vector space $V$ is called *locally unipotent* if $(\phi - 1)^n v = 0$ for all $v \in V$ and some integer $n$, which may depend on $v$.)

We first define a refinement of the filtration $\{\mathcal{P}_n\}$ on $\mathcal{P}$ associated with the given filtration $\mathcal{F}$ on $\mathfrak{g}$. Set $\mathcal{Q}_0 = 0$ and

$$\mathcal{Q}_n = \mathcal{P}_{n-1} + \sum_{0<k<n} \mathcal{P}_k\mathcal{P}_{n-k}. \tag{3.1}$$

It is then immediate that

$$\mathcal{P}_{n-1} \subseteq \mathcal{Q}_n \subseteq \mathcal{P}_n, \tag{3.2}$$

$$\mathcal{P}_m\mathcal{Q}_n \subseteq \mathcal{Q}_{m+n}, \tag{3.3}$$

for $m,n \geqslant 0$.

Considered as functions on the vector space $\mathfrak{g}$, the elements of $\mathcal{Q}_n$ are the *nonlinear* polynomials of filtration weight $\leqslant n$ (modulo $\mathcal{P}_{n-1}$). Indeed, from (2.3) we see that, modulo $\mathcal{P}_{n-1}$, $\mathcal{Q}_n$ is spanned by monomials $\xi^\alpha$ such that $w(\alpha) = n$ and $|\alpha| \geqslant 2$. If the filtration $\mathcal{F}$ is of length $l$, this makes it evident that

$$\mathcal{Q}_n = \mathcal{P}_n \quad \text{when } n > l. \tag{3.4}$$

Now let Der($\mathcal{P}$) and Aut($\mathcal{P}$) denote the derivations and automorphisms, respectively, of the algebra $\mathcal{P}$. We define

$$\mathfrak{m} = \{T \in \mathrm{Der}(\mathcal{P}): T\mathcal{P}_n \subseteq \mathcal{Q}_n, \forall n \geqslant 0\},$$

$$M = \{\phi \in \mathrm{Aut}(\mathcal{P}): (\phi - 1)\mathcal{P}_n \subseteq \mathcal{Q}_n, \forall n \geqslant 0\}.$$

It is easily verified from (3.1)–(3.3) that $\mathfrak{m}$ is a Lie subalgebra of Der($\mathcal{P}$), such that $\mathfrak{m}(\mathcal{Q}_n) \subseteq \mathcal{Q}_n$ and

$$\mathfrak{m}^{n+1}(\mathcal{P}_n) = 0. \tag{3.5}$$

(Here $\mathfrak{m}^n$ denotes the linear span of all products of $n$ elements of $\mathfrak{m}$, considered as linear transformations on $\mathcal{P}$.) It follows from (3.5) that $T \in \mathfrak{m}$ generates an automorphism $\exp(T)$ of $\mathcal{P}$, defined by

$$\exp(T)f = \sum (n!)^{-1} T^n f,$$

for $f \in \mathcal{P}$. Obviously $\exp(\mathfrak{m}) \subseteq M$.

If $\phi, \psi \in M$, then $(\phi\psi - 1)\mathcal{P}_n \subseteq (\phi - 1)\psi\mathcal{P}_n + (\psi - 1)\mathcal{P}_n \subseteq \mathcal{Q}_n$. Also $\phi\mathcal{P}_n = \mathcal{P}_n$, since $\mathcal{P}_n$ is finite-dimensional, and hence

$$(\phi^{-1} - 1)\mathcal{P}_n = (\phi^{-1} - 1)\phi\mathcal{P}_n = (1 - \phi)\mathcal{P}_n \subseteq \mathcal{Q}_n.$$

Thus $M$ is a subgroup of Aut $\mathcal{P}$.

THEOREM 3.1 (a) $\mathfrak{m}$ *is a finite-dimensional nilpotent Lie algebra, and the map* $T \mapsto \exp(T)$ *is a bijection from* $\mathfrak{m}$ *onto* $M$.

(b) *If* $\{\xi_i\}$ *is a basis for* $\mathfrak{g}^*$ *dual to an* $\mathcal{F}$*-basis* $\{X_i\}$, *then* $M$ *consists of all automorphisms of* $\mathcal{P}$ *whose action on* $\{\xi_i\}$ *is of the form*

$$\xi_i \mapsto \xi_i + q_i, \tag{3.6}$$

*with* $q_i \in \mathcal{Q}_{w_i}$ ($w_i$ = $\mathcal{F}$*-weight of* $X_i$).

PROOF. (a) To prove that exp is surjective, let $\phi \in M$ and set $S = \phi - 1$. Then

$$S(\mathcal{P}_n) \subseteq \mathcal{Q}_n, \quad S(\mathcal{Q}_n) \subseteq \mathcal{Q}_n, \quad S(\mathcal{P}_0) = 0.$$

We claim that if $2^m > n$, then

$$S^{m+1}(\mathcal{P}_n) \subseteq \mathcal{P}_{n-1}. \tag{3.7}$$

Indeed, this holds for $n = 1$, since $\mathcal{Q}_1 = \mathcal{P}_0$. By the definition of $\mathcal{Q}_n$, we see that

$$S^m(\mathcal{Q}_n) \subseteq \sum \mathcal{P}_{k_1} \cdots \mathcal{P}_{k_r} + \mathcal{P}_{n-1},$$

where $r = 2^m$ and the summation is over all $k_i > 0$ such that $k_1 + \cdots + k_r = n$. But if $r > n$, this range of summation is empty, which gives (3.7).

Given $S = \phi - 1$ as above, we define a linear transformation $T$ on $\mathcal{P}$ by

$$Tf = \log(\phi)f = \sum_{n \geq 1} \frac{(-1)^{n+1}}{n} S^n f.$$

The range of summation is finite, by (3.7), and $T(\mathcal{P}_n) \subseteq \mathcal{Q}_n$, $T(\mathcal{P}_0) = 0$. By the argument just given, this implies that $T$ is locally nilpotent on $\mathcal{P}$. Hence $\exp T$ is defined as a linear transformation on $\mathcal{P}$, and by the formal power series identity $x = \exp(\log x)$ we have $\exp T = \phi$. It remains to show that $T$ is a derivation of $\mathcal{P}$. For this it suffices to prove that $\exp(tT) \in \mathrm{Aut}(\mathcal{P})$ for all $t \in \mathbf{F}$. But for fixed $f, g \in \mathcal{P}$, the function

$$t \to e^{tT}(fg) - (e^{tT}f)(e^{tT}g)$$

is a polynomial. Since $\exp(nT) = \phi^n$, this polynomial vanishes when $t \in \mathbf{N}$, and hence vanishes identically.

Since $\mathfrak{g}^* \subset \mathcal{P}_l$ ($l$ = length of $\mathcal{F}$), and a derivation of $\mathcal{P}$ is determined by its action on the linear functions, it follows that the representation of $\mathfrak{m}$ on the finite-dimensional subspace $\mathcal{P}_l$ is faithful and nilpotent. Hence $\mathfrak{m}$ is a finite-dimensional nilpotent Lie algebra. To prove that the map $T \to \exp T$ is injective, it thus suffices, by the Campbell-Hausdorff formula, to show that $\exp T = 1$ implies $T = 0$.

If $T \in \mathfrak{m}$ and $f \in \mathcal{P}$, then the function $t \to \exp(tT)f - f$ is a polynomial in $t \in \mathbf{F}$. If $\exp T = 1$, this polynomial vanishes when $t \in \mathbf{N}$, and hence is identically zero. Differentiating at $t = 0$, we conclude that $Tf = 0$, and hence $T = 0$.

(b) By definition, the transformations in $M$ act on $\{\xi_i\}$ as in formula (3.6). Conversely, given $q_i \in \mathcal{Q}_{w_i}$, there is a unique *homomorphism* $\phi$: $\mathcal{P} \to \mathcal{P}$ such that $\phi(\xi_i) = \xi_i + q_i$. Since $q_i$ is a polynomial function of $\{\xi_j: j < i\}$, it is evident that $\phi$ is an automorphism of $\mathcal{P}$. Obviously $\phi(\mathcal{P}_n) = \mathcal{P}_n$ and hence $\phi(\mathcal{Q}_n) = \mathcal{Q}_n$. For $1 \leqslant n \leqslant l$, the linear functions $\{\xi_i: w_i = n\}$ are a complementary basis to $\mathcal{Q}_n$ in $\mathcal{P}_n$. By definition of $\phi$, this implies that $(\phi - 1)\mathcal{P}_n \subseteq \mathcal{Q}_n$ when $1 \leqslant n \leqslant l$. For $n > l$, this also holds, by (3.4), and hence $\phi \in M$. Q.E.D.

REMARK. The constructions of this section are independent of the Lie algebra structure on $\mathfrak{g}$; only the vector space structure and the filtration $\mathcal{F}$ are involved. We shall write $M = M(\mathcal{F})$ when the dependence on $\mathcal{F}$ is to be emphasized. (For a more detailed study of $M$ and related subgroups of $\mathrm{Aut}(\mathcal{P})$, cf. [7, Appendix].)

**4. Canonical coordinate transformations.** Let $\{X_i\}$ be an $\mathcal{F}$-basis for $\mathfrak{g}$. We define a homomorphism $\phi$: $\mathcal{P} \to \mathcal{P}$ by setting

$$\phi(f)(x) = f(p_1(x) * \cdots * p_d(x)), \tag{4.1}$$

where $p_i(x) = \langle \xi_i, x \rangle X_i$ is the projection onto $\mathbf{F}X_i$ and $x * y$ is the Campbell-Hausdorff multiplication on $\mathfrak{g}$, as in §2.

THEOREM 4.1. *Let $M = M(\mathcal{F})$ be the subgroup of* $\mathrm{Aut}(\mathcal{P})$ *defined in* §3. *Then for any $\mathcal{F}$-basis $\{X_i\}$, the transformation $\phi$ defined by* (4.1) *is in $M$.*

REMARKS. 1. The linear functions $\{\xi_i\}$ give "canonical coordinates of the first kind" for the Campbell-Hausdorff group structure on $\mathfrak{g}$, while functions $\eta_i = \phi^{-1}(\xi_i)$ are "canonical coordinates of the second kind," relative to the basis $\{X_i\}$ (cf. [2, III.4.3]). The point of the theorem is that when $\{X_i\}$ is an $\mathcal{F}$-basis, then $\{\eta_i\}$ also generate $\mathcal{P}$, and $\eta_i = \xi_i + q_i$, where $q_i \in \mathcal{Q}_{w_i}$.

2. If $\{X_i\}$ is an arbitrary basis then $\phi$ is not necessarily an automorphism of $\mathcal{P}$. For example, let $\mathfrak{g}$ be the Heisenberg algebra with basis $e_1, e_2, e_3$ and commutation relations $[e_1, e_2] = e_3$, $e_3$ central. Set $X_1 = e_1$, $X_2 = e_2$, $X_3 = e_1 + e_3$, and let $\{\xi_i\}$ be the basis dual to $\{X_i\}$. One finds that the homomorphism $\phi$ defined by (4.1) acts on the generators $\{\xi_i\}$ by

$\phi(\xi_1) = \xi_1 - \frac{1}{2}\xi_2(\xi_1 - \xi_3)$, $\phi(\xi_2) = \xi_2$, and $\phi(\xi_3) = \xi_3 + \frac{1}{2}\xi_2(\xi_1 - \xi_3)$. Thus $\phi(\xi_1 - \xi_3) = (\xi_1 - \xi_3)(1 - \xi_2)$ and $\phi(1 - \xi_2) = 1 - \xi_2$. If $\phi$ were an automorphism of $\mathbf{F}[\xi_1, \xi_2, \xi_3]$, this would give the relation

$$\xi_1 - \xi_3 = h(1 - \xi_2), \tag{4.2}$$

where $h = \phi^{-1}(\xi_1 - \xi_3)$. But no polynomial $h$ can satisfy (4.2), so $\phi \notin \operatorname{Aut}(\mathcal{P})$ in this case.

PROOF OF THEOREM. By the Campbell-Hausdorff formula, we can write

$$p_1(x) * \cdots * p_d(x) = x + C(x),$$

where $C(x)$ is a linear combination of iterated commutators of $\{p_i(x)\}$. Since $\{X_i\}$ is an $\mathcal{F}$-basis, any such iterated commutator is of the form $\xi^\alpha(x)z_\alpha$, where $z_\alpha \in \mathfrak{g}_{w(\alpha)}$, and at least two of the $\alpha_i$ are nonzero. Hence $\xi^\alpha \in \mathcal{Q}_{w(\alpha)}$.

Consider now the action of $\phi$ on the linear functions $\{\xi_i\}$. Since $\xi_i(\mathfrak{g}_n) = 0$ for $n > w_i$, the observations of the previous paragraph imply that $\phi(\xi_i) = \xi_i + q_i$, where $q_i \in \mathcal{Q}_{w_i}$. Part (b) of Theorem 3.1 thus gives $\phi \in M$. Q.E.D.

The next result is the analogue of Corollary 2.2 for the structure constants of the second kind.

THEOREM 4.2. *Let the structure constants $K_\gamma^{\alpha\beta}$ be defined by* (1.2) *relative to an $\mathcal{F}$-basis for $\mathfrak{g}$, and let $\{\xi_i\}$ be the dual basis for $\mathfrak{g}^*$. Define $\eta_i = \phi^{-1}(\xi_i)$, where $\phi$ is the transformation* (4.1), *and set*

$$G_i = (\phi \otimes \phi)\,\mu(\eta_i) \in \mathcal{P} \otimes \mathcal{P}.$$

*Then*

$$K_\gamma^{\alpha\beta} = (\alpha!\beta!)^{-1}\partial^\alpha_{(x)}\partial^\beta_{(y)}G^\gamma(x, y)|_{x=y=0}, \tag{4.3}$$

*where $G^\gamma = G_1^{\gamma_1} \cdots G_d^{\gamma_d}$, and $\mathcal{P} \otimes \mathcal{P}$ is identified with the polynomials on $\mathfrak{g} \times \mathfrak{g}$ as usual.*

PROOF. Let $f \in \mathcal{P}$. Then by the identity

$$e^{x*y} = e^x e^y, \qquad x, y \in \mathfrak{g}$$

(in $[U(\mathfrak{g})]_{\mathcal{P}}$) and the definition of $\phi$, we can express

$$\phi(f) = \sum_\alpha (\alpha!)^{-1}\xi^\alpha\langle f, X^\alpha\rangle.$$

On the other hand, by Taylor's formula (2.6),

$$\phi(f) = \sum_\alpha (\alpha!)^{-1}\xi^\alpha\langle \phi(f), X(\alpha)\rangle.$$

Comparing these two expansions gives the following formula for the automorphism $\phi$, in terms of the duality between $\mathcal{P}$ and $U(\mathfrak{g})$:

$$\langle \phi(f), X(\alpha)\rangle = \langle f, X^\alpha\rangle. \tag{4.4}$$

Now set $\eta_i = \phi^{-1}(\xi_i)$ and $\eta^\alpha = \eta_1^{\alpha_1} \cdots \eta_d^{\alpha_d}$, for $\alpha \in \mathbf{N}^d$. Then by (4.4) and

(2.2) we find that

$$\langle \eta^\alpha, X^\beta \rangle = \alpha!\delta_{\alpha\beta}.$$

Hence the dual form of the defining equation for the structure constants $K_\gamma^{\alpha\beta}$ is

$$\mu(\eta^\gamma) = \sum_{\alpha,\beta} K_\gamma^{\alpha\beta}\eta^\alpha \otimes \eta^\beta \tag{4.5}$$

(the range of summation is finite, by Proposition 1.1). Applying the automorphism $\phi \otimes \phi$ of $\mathcal{P} \otimes \mathcal{P}$ to this equation, we thus obtain the expansion

$$G^\gamma = \sum_{\alpha,\beta} K_\gamma^{\alpha\beta}\xi^\alpha \otimes \xi^\beta, \tag{4.6}$$

which is equivalent to (4.3). Q.E.D.

The transformation between the bases of the first and second kind for $U(\mathfrak{g})$ can be expressed in terms of the action of $M$ as follows:

THEOREM 4.3. *Let $C_\alpha^\beta$ and $D_\alpha^\beta$ be defined by the equations*

$$\begin{aligned} (\alpha!)^{-1}X^\alpha &= \sum_\beta (\beta!)^{-1}C_\alpha^\beta X(\beta), \\ (\alpha!)^{-1}X(\alpha) &= \sum_\beta (\beta!)^{-1}D_\alpha^\beta X^\beta, \end{aligned} \tag{4.7}$$

*relative to an $\mathcal{F}$-basis $\{X_i\}$ for $\mathfrak{g}$. If $\{\xi_i\}$ is the dual basis for $\mathfrak{g}^*$, then*

$$\begin{aligned} C_\alpha^\beta &= (\alpha!)^{-1}\partial^\alpha\phi(\xi^\beta)\big|_{x=0}, \\ D_\alpha^\beta &= (\alpha!)^{-1}\partial^\alpha\phi^{-1}(\xi^\beta)\big|_{x=0}, \end{aligned} \tag{4.8}$$

*where $\phi$ is defined by* (4.1). *In particular,*

$$D_\alpha^\beta = C_\alpha^\beta = 0 \quad \textit{if } |\alpha| < |\beta| \textit{ or } w(\alpha) > w(\beta). \tag{4.9}$$

REMARK. By the PBW theorem, the constants $C_\alpha^\beta$ and $D_\alpha^\beta$ are determined by (4.7). For a recursive procedure for calculating these constants for any Lie algebra, cf. **[10]**.

PROOF. Using equation (4.4), we find that

$$C_\alpha^\beta = (\alpha!)^{-1}\langle \xi^\beta, X^\alpha \rangle = (\alpha!)^{-1}\langle \phi(\xi^\beta), X(\alpha)\rangle,$$

and similarly,

$$D_\alpha^\beta = (\alpha!)^{-1}\langle \phi^{-1}(\xi^\beta), X(\alpha)\rangle.$$

By (2.2) this gives (4.8). Since $\phi \in M$, we have

$$\phi^{-1}(\xi^\beta), \phi(\xi^\beta) \in \mathcal{P}_{w(\beta)}.$$

Hence the vanishing condition (4.9) when $w(\alpha) > w(\beta)$ follows from (4.8) and (2.3). The vanishing condition when $|\alpha| < |\beta|$ follows by using the canonical filtration of $U(\mathfrak{g})$ [3, Proposition 2.4.4].

**5. Estimates for structure constants.** Assume now that the scalar field $\mathbf{F} = \mathbf{C}$. With the same hypotheses and notation as in Corollary 2.2 and Theorems 4.2 and 4.3, we have the following estimate for the order of growth of the structure constants for $U(\mathfrak{g})$ associated with an $\mathfrak{F}$-basis for $\mathfrak{g}$:

THEOREM 5.1. *There is a constant* $R \geq 1$ *such that*

$$\max_{\alpha,\beta}\left\{\left|C_\gamma^{\alpha\beta}\right|, \left|K_\gamma^{\alpha\beta}\right|, \left|C_\gamma^{\alpha}\right|, \left|D_\gamma^{\alpha}\right|\right\} \leq R^{|\gamma|}. \tag{5.1}$$

PROOF. Equation (2.7) and the Cauchy estimates in a polydisc give the inequalities

$$\left|C_\gamma^{\alpha\beta}\right| \leq \max\left\{\left|F^\gamma(x,y)\right| : x, y \in D\right\}, \tag{5.2}$$

where $D = \{x \in \mathfrak{g}: |\xi_i(x)| = 1 \text{ for } 1 \leq i \leq d\}$. The right side of (5.2) is majorized in turn by $R^{|\gamma|}$, provided we take

$$R \geq \max\left\{\left|F_i(x,y)\right| : x, y \in D, 1 \leq i \leq d\right\}.$$

The same argument applies to the other structure constants, using formulas (4.3) and (4.8).

**6. Locally convex completions of $U(\mathfrak{g})$.** We continue to assume that the scalar field $\mathbf{F} = \mathbf{C}$. We shall use the estimates of §5 to construct complete, locally-convex topological algebras $\mathcal{A}$ such that

$$U(\mathfrak{g}) \subset A \subset \left[U(\mathfrak{g})\right]_{\mathfrak{F}},$$

with $U(\mathfrak{g})$ dense in $A$.

Fix an $\mathfrak{F}$-basis $\{X_i\}$ for $\mathfrak{g}$. Recall that the numbers $w_i = w(X_i)$ are uniquely determined by the filtration $\mathfrak{F}$.

DEFINITION. A sequence $\mathfrak{M} = \{M_\alpha: \alpha \in \mathbf{N}^d\}$ of positive numbers is an *$\mathfrak{F}$-weight sequence* if $M_0 = 1$ and

$$M_\gamma \leq M_\alpha M_\beta \tag{6.1}$$

when $w(\gamma) \geq w(\alpha) + w(\beta)$ (where $w(\alpha) = \sum w_i\alpha_i$).

EXAMPLE. Take $M_\alpha = \phi(w(\alpha))$, where $\phi$ is defined on $\mathbf{N}$ and satisfies $\phi(0) = 1$, $\phi(m) > 0$, and $\phi(m + n) \leq \phi(m)\phi(n)$. For instance, let $\phi(m) = m^{-pm}$, with $p$ any nonnegative real number (note that $m^m n^n \leq (m+n)^{m+n}$ by the geometric-arithmetic mean inequality).

Assume now that $\mathfrak{M} = \{M_\alpha\}$ is an $\mathfrak{F}$-weight sequence. Define a family of seminorms on $[U(\mathfrak{g})]_{\mathfrak{F}}$ as follows:

If $T = \sum_\alpha (\alpha!)^{-1} c_\alpha X^\alpha$ and $r > 0$, then set

$$\|T\|_r = \sup_\alpha \{ r^{|\alpha|} M_\alpha |c_\alpha| \} \tag{6.2}$$

(the value $+\infty$ is allowed for $\|T\|_r$).

LEMMA 6.1. *Let $d = \dim \mathfrak{g}$, $l$ = length of $\mathfrak{F}$, and let $R$ be the constant in Theorem 5.1. Then for $p > 2dR \max[r, r^{1/l}]$, there is a constant $C = C(d, l, p, r, R) < \infty$ such that*

$$\|ST\|_r \leqslant C \|S\|_p \|T\|_p \tag{6.3}$$

*for all $S, T \in [U(\mathfrak{g})]_{\mathfrak{F}}$.*

PROOF. By Remark 1 after Proposition 1.2, $\|ST\|_r$ is majorized by

$$\sup_\gamma \left\{ r^{|\gamma|} M_\gamma \sum_{\alpha,\beta} |a_\alpha b_\beta K_\gamma^{\alpha\beta}| \right\}, \tag{6.4}$$

if $S = \Sigma_\alpha (\alpha!)^{-1} a_\alpha X^\alpha$ and $T = \Sigma_\beta (\beta!)^{-1} b_\beta X^\beta$. From Theorem 5.1 and the definition (6.2), we find that (6.4) is majorized by

$$\sup_\gamma \left\{ \sum \frac{M_\gamma}{M_\alpha M_\alpha} (rR)^{|\gamma|} p^{-|\alpha|-|\beta|} \right\} \|S\|_p \|T\|_p. \tag{6.5}$$

In (6.5) the summation is over $\alpha, \beta \in \mathbf{N}^d$ such that $|\alpha| + |\beta| \geqslant |\gamma|$ and $w(\alpha) + w(\beta) \leqslant w(\gamma)$, by virtue of Proposition 1.1. Since $|\alpha| \leqslant w(\alpha) \leqslant l|\alpha|$, this range of summation is contained in the set

$$E_\gamma = \{ \alpha, \beta \in \mathbf{N}^d \colon |\gamma| \leqslant |\alpha| + |\beta| \leqslant l|\gamma| \}.$$

Since $M_\gamma \leqslant M_\alpha M_\beta$ in (6.5), this gives (6.3), where

$$C = \sup_\gamma \left\{ (rR)^\gamma \sum_{E_\gamma} p^{-|\alpha|-|\beta|} \right\}.$$

The finiteness of $C$ when $p > 2dR \max[r, r^{1/l}]$ follows from elementary estimates.

LEMMA 6.2. *Let $\{\xi_i\}$ be the dual basis to $\{X_i\}$. Then if $s > dR \max[r, r^{1/l}]$, there is a constant $C_1 < \infty$ such that*

$$\|T\|_r \leqslant C_1 \sup_\alpha \{ s^{|\alpha|} M_\alpha |\langle T, \xi^\alpha \rangle| \}. \tag{6.6}$$

*Conversely, if $r > dR \max[s, s^{1/l}]$, then there is a constant $C_2 < \infty$ such that*

$$\sup_\alpha \{ s^{|\alpha|} M_\alpha |\langle T, \xi^\alpha \rangle| \} \leqslant C_2 \|T\|_r. \tag{6.7}$$

PROOF. Given $s > 0$ and $T \in [U(\mathfrak{g})]_{\mathfrak{F}}$, we define

$$|||T|||_s = \sup_\alpha \{ s^{|\alpha|} M_\alpha |\langle T, \xi^\alpha \rangle| \}.$$

In order to compare the two families of seminorms $\{\|T\|_r\}$ and $\{|||T|||_s\}$, we note that if $T$ has the formal series expansion

$$T = \sum_{\alpha} (\alpha!)^{-1} b_\alpha X(\alpha),$$

then $b_\alpha = \langle T, \xi^\alpha \rangle$. It follows that $\|T\|_r \leqslant D(r, s) |||T|||_s$, where

$$D(r, s) = \sum_{\alpha} (\alpha!)^{-1} s^{-|\alpha|} M_\alpha^{-1} \|X(\alpha)\|_r.$$

Now from the definition, we have $(\beta!)^{-1}\|X^\beta\|_r = r^{|\beta|}M_\beta$. Using Theorems 4.3 and 5.1, we can thus estimate

$$(\alpha!)^{-1}\|X(\alpha)\|_r \leqslant \sum_{\beta} (\beta!)^{-1} |D_\alpha^\beta| \, \|X\|_r \leqslant R^{|\alpha|} \sum r^{|\beta|} M_\beta,$$

with the last summation over $\beta \in \mathbf{N}^d$ such that $|\beta| \leqslant |\alpha|$ and $w(\beta) \geqslant w(\alpha)$. In this range $M_\beta \leqslant M_0 M_\alpha = M_\alpha$, and $|\alpha|/l \leqslant |\beta| \leqslant |\alpha|$, so this gives the bound

$$D(r, s) \leqslant \sum (R/s)^{|\alpha|} r^{|\beta|}$$

(sum over $\alpha, \beta$ with $|\alpha|/l \leqslant |\beta| \leqslant |\alpha|$). Elementary estimates show that $D(r, s) < \infty$, provided that $s > dR \max[r, r^{1/l}]$. This proves (6.6).

To obtain an estimate in the opposite direction, we write

$$T = \sum_{\beta} (\beta!)^{-1} a_\beta X^\beta = \sum_{\alpha,\beta} (\alpha!)^{-1} a_\beta C_\beta^\alpha X(\alpha),$$

where $\{C_\beta^\alpha\}$ are the constants defined by equations (4.7). Thus

$$\langle T, \xi^\alpha \rangle = \sum_{\beta} C_\beta^\alpha a_\beta,$$

so that

$$|\langle T, \xi^\alpha \rangle| \leqslant C(\alpha, r)\|T\|_r,$$

where

$$C(\alpha, r) = \sum_{\beta} |C_\beta^\alpha| r^{-|\beta|} M_\beta^{-1}. \tag{6.8}$$

By Theorem 4.3 we can restrict $\beta$ to the range $|\alpha| \leqslant |\beta|$ and $w(\beta) \leqslant w(\alpha)$ in the summation (6.8). Hence $M_\alpha \leqslant M_0 M_\beta = M_\beta$, so that

$$C(\alpha, r) \leqslant M_\alpha^{-1} \sum (R/r)^{|\beta|}$$

(sum over $|\alpha| \leqslant |\beta| \leqslant l|\alpha|$). From this estimate (6.7) follows easily, completing the proof.

We can now state the main result of this section.

THEOREM 6.3. *Let $\mathfrak{M}$ be an $\mathfrak{F}$-weight sequence, and let the seminorms $\|T\|_r$ be defined by* (6.2) *relative to some $\mathfrak{F}$-basis for* $\mathfrak{g}$. *Define*

(6.9) $$A = \{T \in [U(\mathfrak{g})]_{\mathfrak{F}} \colon \|T\|_r < \infty \text{ for all } r > 0\}.$$

*Then*

(1) *$A$ is a subalgebra of $[U(\mathfrak{g})]_{\mathfrak{F}}$*;

(2) *With the locally convex topology defined by the family of seminorms $\{\|\cdot\|_r\}_{r>0}$, $A$ is a Fréchet space containing $U(\mathfrak{g})$ as a dense subspace, and multiplication is jointly continuous*;

(3) *$A$ and its topology are independent of the choice of $\mathfrak{F}$-basis used to define the seminorms $\|\cdot\|_r$.*

DEFINITION. If $\mathfrak{M}$ is an $\mathfrak{F}$-weight sequence, then the *$\mathfrak{M}$-completion* $[U(\mathfrak{g})]_{\mathfrak{M}}$ of $U(\mathfrak{g})$ is the topological algebra $A$ of Theorem 6.3.

PROOF OF THEOREM. Statements (1) and (2) are easy consequences of Lemma 6.1 and straightforward estimates. For example, to show that $U(\mathfrak{g})$ is dense in $A$, let $T \in A$ have the formal expansion

$$T = \sum_{\alpha} (\alpha!)^{-1} b_\alpha X^\alpha,$$

and define $T_n$ by the same series, but with $|\alpha| < n$. Now $|b_\alpha| \leq s^{-|\alpha|} M_\alpha^{-1} \|T\|_s$, and $\|X^\alpha\|_r = \alpha! M_\alpha r^{|\alpha|}$, for any $r, s > 0$. Thus if $s > r$, we can estimate $\|T - T_n\|_r \leq C_n \|T\|_s$, where

$$C_n = \sum_{|\alpha| > n} (r/s)^{|\alpha|}.$$

Since $C_n \to 0$ as $n \to \infty$, this shows that $T_n \to T$ in the $A$-topology.

To prove (3), let $\{X_i\}$ and $\{Y_i\}$ be two $\mathfrak{F}$-bases for $\mathfrak{g}$, with corresponding dual bases $\{\xi_i\}$ and $\{\eta_i\}$. Then $w(X_i) = w(Y_i) = w_i$ and $X_i = \sum c_{ij} Y_j$, with $c_{ij} = 0$ if $w_j < w_i$. It follows that for $\alpha \in \mathbf{N}^d$ we can express

$$\eta^\alpha = \sum g_{\alpha\beta} \xi^\beta,$$

where the summation is over $\beta \in \mathbf{N}^d$ such that $w(\beta) \leq w(\alpha)$ and $|\beta| = |\alpha|$. Furthermore, $g_{\alpha\beta}$ is a product of $|\alpha|$ factors $c_{ij}$, so that $|g_{\alpha\beta}| \leq R^{|\alpha|}$, for some constant $R$. Thus we have

(6.10) $$|\langle T, \eta^\alpha \rangle| \leq (Rd)^{|\alpha|} \max_{\substack{w(\beta) \leq w(\alpha) \\ |\beta| = |\alpha|}} |\langle T, \xi^\beta \rangle|.$$

Using (6.10) and Lemma 6.2, it is then easy to verify that the two $\mathfrak{F}$-bases give equivalent families of seminorms on $[U(\mathfrak{g})]_{\mathfrak{F}}$. Q.E.D.

REMARKS. (1) It follows by Lemma 6.2 that the canonical symmetrization map [3, §2.4.5]:

$$\omega \colon S(\mathfrak{g}) \to U(\mathfrak{g})$$

extends to a linear isomorphism between $[S(\mathfrak{g})]_{\mathfrak{M}}$ and $[U(\mathfrak{g})]_{\mathfrak{M}}$. Here we define $[S(\mathfrak{g})]_{\mathfrak{M}} = [U(V)]_{\mathfrak{M}}$, where $V$ is the vector space $\mathfrak{g}$ but with trivial Lie bracket.

(2) By the same argument as in the proof of Theorem 6.3, one sees that if $\phi \in \mathrm{Aut}(\mathfrak{g})$ and $\phi(\mathfrak{g}_n) \subseteq \mathfrak{g}_n$, then $\phi$ extends by continuity from $U(\mathfrak{g})$ to $[U(\mathfrak{g})]_{\mathfrak{M}}$.

(3) Combining Remarks 1 and 2, we conclude that the adjoint representation of $G$ on $U(\mathfrak{g})$ extends by continuity to a representation on $[U(\mathfrak{g})]_{\mathfrak{M}}$, which is equivalent to the representation on $[S(\mathfrak{g})]_{\mathfrak{M}}$. (Here $G = \exp \mathfrak{g}$.) In this respect the algebras $[U(\mathfrak{g})]_{\mathfrak{M}}$ are more satisfactory than the completions $\mathcal{Q}_\lambda$, $\lambda < 1$, constructed in [5].

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